Online Appendix

Monopolistic Competition and Optimum Product Diversity Under Firm Heterogeneity

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July 4, 2014

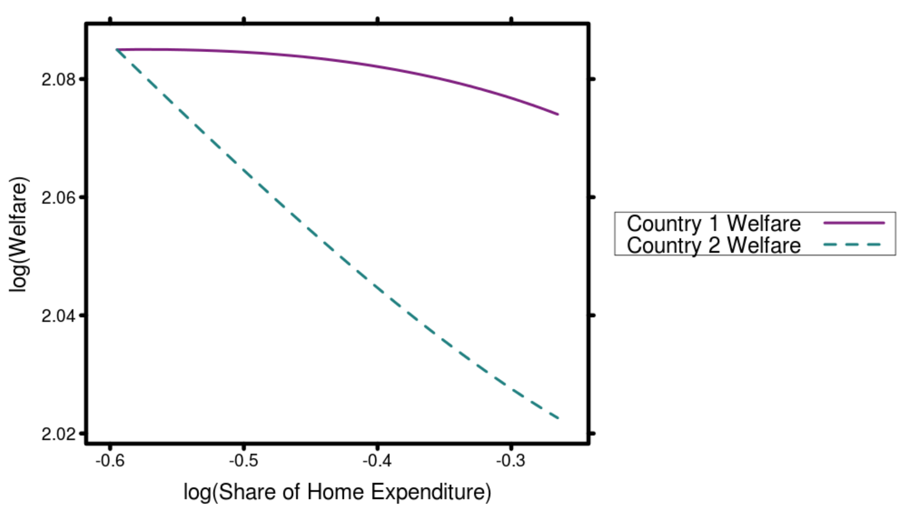
1 VES Specific Utility

The VES demand system implied by $u(q) = aq^{\rho} + bq^{\gamma}$ can generate all four combinations of increasing and decreasing, private and social markups as we now briefly discuss. First, note that

$$\varepsilon'(q) = ab(\rho-\gamma)^2 q^{\rho-\gamma-1} / \left(aq^{\rho-\gamma}+b\right)^2,$$
$$\mu'(q) = -ab\rho\gamma(\rho-\gamma)^2 q^{\rho-\gamma-1} / \left(a\rho q^{\rho-\gamma}+b\gamma\right).$$

For $\rho = \gamma$, $\varepsilon'(q) = \mu'(q) = 0$ and we are in a CES economy. For $\rho \neq \gamma$, sign $\varepsilon'(q) =$ sign ab and sign $\mu'(q) =$ sign $-ab \cdot \rho\gamma$, exhibiting all four combinations for appropriate parameter values. In addition, this demand system does not exhibit the log-linear relationship between welfare and share of expenditure on home goods discussed in Arkolakis et al. (2012), as shown in Figure 1 for $u(q) = q^{1/2} + q^{1/4}$.

Figure 1: Welfare and Share of Home Expenditure as Home Tariff Increases



2 Converse of the Folk Theorem

We now consider general consumer preferences of the form given by Equation (1).

$$U(M_e, c_d, q) \equiv \nu(M_e, c_d) \int_0^{c_d} u(q(c))g(c)dc \tag{1}$$

where ν is positive and continuously differentiable, and u satisfies Definition 1.

Proposition. *Under VES demand, a necessary condition for the market equilibrium to be socially optimal is that u is CES.*

Proof. Assume an equilibrium exists which is socially optimal with M_e and c_d fixed by that equilibrium. Also let $q^*(c)$ denote equilibrium quantities. If the equilibrium is efficient for these fixed M_e and c_d, the quantities $q_p(c)$ a policymaker would choose must be optimal. For convenience, define the functional $H(q)$ as in the above proof and let $U^*(q) \equiv U(M_e, c_d, q)$ be as in Equation (1). By Theorems 5.11 and 5.15 of Troutman, a necessary condition for q_p to be optimal is that either $\delta H(q_p;\xi) = 0\ \forall \xi \in \mathscr{C}^1[0, c_d]$ or $\exists \lambda$ s.t. $\delta U^*(q_p) = \lambda \delta H(q_p;\xi) = 0$ $\forall \xi \in \mathscr{C}^1[0, c_d]$. We will rule out the first and exploit an implication of the second.
Case 1: $\delta H(q_p;\xi) = 0\ \forall \xi \in \mathscr{C}^1[0, c_d]$. $\forall \xi$ we have that

$$\delta H(q_p;\xi) = \int_0^{c_d} \xi(c)cg(c)dc = 0$$

which implies $cg(c)$ is identically zero on $[0, c_d]$ which is clearly not optimal.
Case 2: $\delta U^*(q_p) = \lambda \delta H(q_p;\xi)\ \forall \xi \in \mathscr{C}^1[0, c_d]$. For any fixed M_e and c_d and $\forall \xi$ we have that

$$\nu(M_e, c_d) \int_0^{c_d} \xi(c)u'(q_p(c))g(c)dc = \lambda M_e \int_0^{c_d} \xi(c)cg(c)dc$$

so for $\lambda' \equiv \lambda M_e/\nu(M_e, c_d)$ we have $\int_0^{c_d}[u'(q_p(c)) - \lambda' c]g(c)\xi(c)dc = 0$ and since g is $\mathscr{C}^1$ and strictly positive, we conclude

$$u'(q_p(c)) = \lambda' c \tag{2}$$

Using similar reasoning, a monopolist with costs c picks $q_m(c)$ according to

$$\max_{q_m(c)} [D(q_m(c)) - c]q_m(c) = \max_{q_m(c)} [u'(q_m(c))/\delta - c]q_m(c) \tag{Market}$$

so long as the resulting profit covers f. By assumption, the quantity FOC $[u'(q_m(c))/\delta - c] + u''(q_m(c))q_m(c)/\delta = 0$ uniquely determines each monopolist's optimal quantity which must be $q^*(c)$ in equilibrium. We conclude that $q^*(c)$ is implicitly determined by the monopolist FOC as given in Equation (3).

$$u'(q^*(c)) + u''(q^*(c))q^*(c) = \delta c \tag{3}$$

We now show $q^* = q_p$. Since $H(q_p) = H(q^*)$ and $H(q)$ is linear in q, any convex combination $q_\alpha \equiv \alpha q^* + (1-\alpha)q_p$ has $H(q_\alpha) = H(q_p) = H(q^*)$ and so is attainable. Since u is strictly concave, a standard concavity argument shows that the optimality of q_p and q^* implies $q_p = q_\alpha = q^* \quad \forall \alpha \in [0,1]$. Now comparing Equations (2) and (3) with the knowledge that $q^* = q_p$ and dividing the second by the first we see Equation (4) holds on $[0, c_d]$.

$$1 + u''(q_p(c))q_p(c)/u'(q_p(c)) = \delta/\lambda' \tag{4}$$

Equation (4) implies for some constant k_0 that for each $c \in [0, c_d]$ that

$$u''(q_p(c))q_p(c) \quad = \quad k_0 u'(q_p(c))$$

Equation (3) paired with $u'' < 0$ shows that $q(c)$ is strictly decreasing so we have that $q([0,c_d]) = [q(c_d), q(0)]$. Consequently, $\forall x \in [q(c_d), q(0)]$ we have that $u''(x)x = k_0 u'(x)$. Standard solution techniques imply that the unique continuously differentiable solution for u on $[0, c_d]$ is $u(x) = \alpha + \beta x^\gamma$ for constants α, β, γ, which is precisely the CES form up to an affine transformation. □

3 Trade and Market Size

Proposition. *Free trade between countries of sizes $L_1, \ldots, L_n$ has the same market outcome as a unified market of size $L = L_1 + \ldots + L_n$.*

Proof. Consider a home country of size L opening to trade with a foreign country of size L^*. Suppose the consumer's budget multipliers are equal in each country so $\delta = \delta^*$ and that the terms of trade are unity. We will show that the implied allocation can be supported by a set of prices and therefore constitutes a market equilibrium. The implied quantity allocation, productivity level and per capita entry are the same across home and foreign consumers, so opening to trade is equivalent to an increase in market size from L to $L + L^*$.

Let e denote the home terms of trade, so

$$e \equiv M_e^* \int_0^{c_d^*} p_x^* q_x^* L dG / M_e \int_0^{c_d} p_x q_x L^* dG$$

and by assumption $e = e^* = 1$. Then the $MR = MC$ condition implies a home firm chooses $p(c)[1-\mu(q(c))] = c$ in the home market and $e \cdot p_x(c)[1-\mu(q_x(c))] = c$ in the foreign market. A foreign firm chooses $e^* \cdot p^*(c)[1-\mu(q^*(c))] = c$ in the foreign market and $p_x^*(c)[1-\mu(q_x^*(c))] = c$ in the home market. When $\delta = \delta^*$ and $e = e^* = 1$, quantity allocations and prices are identical, i.e. $q(c) = q_x^*(c) = q^*(c) = q_x(c)$ and $p(c) = p_x^*(c) = p^*(c) = p_x(c)$.

This implies cost cutoffs are also the same across countries. The cost cutoff condition for home firms is $\pi + e\pi_x = (p(c_d) - c_d)\,q(c_d)L + e\,(p_x(c_d) - c_d)\,q_x(c_d)L^* = f$. Substituting for optimal q^* and q_x^* in the analogous foreign cost cutoff condition implies $c_d = c_d^*$. From the resource constraint, this fixes the relationship between entry across countries as $L/M_e = \int_0^{c_d}[cq(c) + cq_x(c) + f]dG + f_e = L^*/M_e^*$. Thus, $\delta = \delta^*$ and $e = e^* = 1$ completely determines the behavior of firms. What remains is to check that $\delta = \delta^*$ and $e = e^* = 1$ is consistent with the consumer's problem and the balance of trade at these prices and quantities consistent with firm behavior.

For the consumer's problem, we require at home that $1 = M_e \int_0^{c_d} pqdG + M_e^* \int_0^{c_d^*} p_x^* q_x^* dG$, which from $L/M_e = L^*/M_e^*$ is equivalent to

$$L/M_e = L\int_0^{c_d} pqdG + L^*\int_0^{c_d^*} p_x^* q_x^* dG = L\int_0^{c_d} pqdG + L/M_e - L\int_0^{c_d} p_x q_x dG.$$

Therefore to show the consumer's problem is consistent, it is sufficient to show expenditure on home goods is equal to expenditure on exported goods ($\int_0^{c_d} pqdG = \int_0^{c_d} p_x q_x dG$), which indeed holds by the above equalities of prices and quantities. To show the balance of trade is consistent, we use the consumer budget constraint which gives

$$e = M_e^* \int_0^{c_d^*} p_x^* q_x^* L dG / M_e \int_0^{c_d} p_x q_x L^* dG = M_e^* L / M_e L^* = 1.$$

Similarly, the implied foreign terms of trade is $e^* = 1$. Thus $\delta = \delta^*$ and $e = e^* = 1$ generate an allocation consistent with monopolistic competition and price system consistent with consumer maximization and free trade. □

4 Large Market Results

Lemma. *As market size becomes large:*

1. *Market revenue is increasing in market size and goes to infinity.*

2. *At the optimum, utility per capita is increasing in market size and goes to infinity.*

3. *Market entry goes to infinity.*

Proof. From above, the market allocation solves

$$\max_{M_e, c_d, q(c)} LM_e \int_0^{c_d} u'(q(c))\, q(c) dG \text{ subject to } L \geq M_e \left(\int_0^{c_d} Lcq(c) + f dG + F_e \right).$$

Let $R(L) \equiv M_e \int_0^{c_d} u'(q(c))\, q(c) dG$ be the revenue per capita under the market allocation. Fix L and let $\{q(c), c_d, M_e\}$ denote the market allocation with L resources. Consider an increased resource level $\tilde{L} > L$ with allocation $\{\tilde{q}(c), \tilde{c}_d, \tilde{M}_e\} \equiv \{(L/\tilde{L}) \cdot q(c), c_d, (\tilde{L}/L) \cdot M_e\}$ which direct inspection shows is feasible. This allocation generates revenue per capita of

$$\tilde{M}_e \int_0^{\tilde{c}_d} u'(\tilde{q}(c))\, q(c) dG = M_e \int_0^{c_d} u'\left((L/\tilde{L}) \cdot q(c)\right) q(c) dG \leq R(\tilde{L}).$$

Since u is concave, it follows that $R(\tilde{L}) > R(L)$. Since $\tilde{q}(c) = (L/\tilde{L}) \cdot q(c) \longrightarrow 0$ for all $c > 0$ and $\lim_{q \longrightarrow 0} u'(q) = \infty$, revenue per capita goes to infinity as $\tilde{L} \longrightarrow \infty$. A similar argument holds for the social optimum.

First note that $q(c)$ is fixed by $u'(q(c))\,[1 - \mu(q(c))] = \delta c$, and $\delta \longrightarrow \infty$ and $\mu(q(c))$ is bounded, it must be that $u'(q(c)) \longrightarrow \infty$ for $c > 0$. This requires $q(c) \longrightarrow 0$ for $c > 0$. Since revenue $u'(q(c))\, q(c)$ is equal to $\varepsilon(q(c))\, u(q(c))$ and ε is bounded, revenue also goes to zero for each $c > 0$. Revenue is also decreasing in δ for every c, so we can bound revenue with a function $B(c)$. In particular, for any fixed market size $\tilde{L}$ and implied allocation $\{\tilde{q}(c), \tilde{c}_d, \tilde{M}_e\}$, for $L \geq \tilde{L}$:

$$u'(q(c))\, q(c) \mathbf{1}_{[0,c_d]}(c) \leq u'(\tilde{q}(c))\, \tilde{q}(c) \mathbf{1}_{[0,\tilde{c}_d]}(c) + u'(\tilde{q}(\tilde{c}_d))\, \tilde{q}(\tilde{c}_d) \mathbf{1}_{[\tilde{c}_d,\infty]}(c) \equiv B(c) \tag{5}$$

where we appeal to the fact that $q(c)$ is decreasing in c for any market size. Since for any L, $\int_0^{c_d} u'(q(c))\, q(c) dG = \delta / M_e$, it is clear that $\int_0^{\infty} B(c) dG = \int_0^{\tilde{c}_d} u'(\tilde{q}(c))\, \tilde{q}(c) dG + u'(\tilde{q}(c_d))\, \tilde{q}(c_d) < \infty$. Since $u'(q(c))\, q(c)$ converges pointwise to zero for $c > 0$, we conclude

$$\lim_{L \longrightarrow \infty} \int_0^{c_d} u'(q(c))\, q(c) dG = \int_0^{c_d} \lim_{L \longrightarrow \infty} u'(q(c))\, q(c) dG = 0$$

by dominated convergence. Therefore $\lim_{L \longrightarrow \infty} \delta / M_e = 0$ which with $\delta \longrightarrow \infty$ shows $M_e \longrightarrow \infty$. The optimal allocation case is similar. □

Lemma. *For all market sizes and all positive marginal cost ($c > 0$) firms:*

1. *Profits ($\pi(c)$) and social profits ($\varpi(c) \equiv (1 - \varepsilon(c)) / \varepsilon(c) \cdot cq(c)L - f$) are bounded.*

2. *Total quantities ($Lq(c)$) in the market and optimal allocation are bounded.*

Proof. For any costs $c_L < c_H$, $q(c_H)$ is in the choice set of a firm with costs c_L and therefore

$$\pi(c_L) \geq (p(c_H) - c_L)\, q(c_H)L - f = \pi(c_H) + (c_H - c_L)\, q(c_H)L. \tag{6}$$

Furthermore, for every $\tilde{c} > 0$, we argue $\pi(\tilde{c})$ is bounded. For $\underline{c} \equiv \tilde{c}/2$, $\pi(\tilde{c}) \leq \pi(\underline{c})$ while $\pi(\underline{c})$ is bounded since $\lim_{L\to\infty} \int_0^{c_d} \pi(c)dG = F_e$ and $\limsup_{L\to\infty} \pi(\underline{c}) = \infty$ would imply $\limsup_{L\to\infty} \int_0^{c_d} \pi(c)dG = \infty$. It follows from Equation (6) that $Lq(c)$ is bounded. Substituting ϖ for π leads to similar arguments for the social optimum. □

Lemma. *Assume interior convergence. Then as market size grows large:*

1. *In the market, $p(c)$ converges in $(0,\infty)$ for $c > 0$ and $Lq(c_d)$ converges in $(0,\infty)$.*
2. *In the optimum, $u \circ q(c)/\lambda q(c)$ converges in $(0,\infty)$ for $c > 0$, $Lq(c_d)$ converges in $(0,\infty)$.*

Proof. Since $q(c) \longrightarrow 0$ for all $c > 0$, $\lim_{q\to 0} \mu(q) \in (0,1)$ shows $\lim_{L\to\infty} p(c)$ aligns with constant markups and thus converges for all $c > 0$. In particular, $p(c_d)$ converges and $L\,(p(c_d) - c_d)\, q(c_d) = f$ so it follows $Lq(c_d)$ converges. Similar arguments hold for the social optimum. □

Lemma. *Assume interior convergence and large market identification. Then for the market and social optimum, $Lq(c)$ converges for $c > 0$.*

Proof. Fix any $c > 0$ and first note that for both the market and social planner, $q(c)/q(c_d) = Lq(c)/Lq(c_d)$ and both $Lq(c)$ and $Lq(c_d)$ are bounded, so $q(c)/q(c_d)$ is bounded.

Now consider the market. $q(c)/q(c_d) \geq 1$ has at least one limit point and if it has two limit points, say a and b with $a < b$, there exist subsequences $(q(c)/q(c_d))_{a_n} \to a$ and $(q(c)/q(c_d))_{b_n} \to b$. There also exist distinct κ and $\tilde{\kappa}$ in (a,b) so that eventually

$$(q(c))_{a_n} < \kappa q(c_d)_{a_n} < \tilde{\kappa} q(c_d)_{b_n} < (q(c))_{b_n}.$$

With $u'' < 0$ this implies

$$\begin{aligned}\left(u'(q(c))/u'(q(c_d))\right)_{a_n} &> \left(u'(\kappa q(c_d))/u'(q(c_d))\right)_{a_n} > \left(u'(\tilde{\kappa} q(c_d))/u'(q(c_d))\right)_{b_n} \\ &> \left(u'(q(c))/u'(q(c_d))\right)_{b_n}.\end{aligned}$$

By assumption, $\lim_{q\to 0} u'(\kappa q)/u'(q) > \lim_{q\to 0} u'(\tilde{\kappa} q)/u'(q)$ but since $q(c) \longrightarrow 0$,

$$\begin{aligned}
\lim_{n\to\infty}\left(u'\circ q(c)/u'\circ q(c_d)\right)_{a_n} &= \lim_{n\to\infty}\left([1-\mu\circ q(c)]\,c/[1-\mu\circ q(c_d)]\,c_d\right)_{a_n} = c/c_d \\
&= \lim_{n\to\infty}\left(u'\circ q(c)/u'\circ q(c_d)\right)_{b_n}
\end{aligned}$$

where we have used the fact that $\lim_{q\to 0}\mu(q) \in (0,1)$, however by assumption this contradicts $a < b$.

For the social optimum, this argument holds (substituting $\varepsilon \neq 0$ for $u'' < 0$) so long as

$$\kappa \neq \tilde{\kappa} \text{ implies } \lim_{q\to 0}\left(u(\kappa q)/\kappa q\right)/\left(u(q)/q\right) \neq \lim_{q\to 0}\left(u(\tilde{\kappa} q)/\kappa q\right)/\left(u(q)/q\right). \tag{7}$$

Since $\lim_{q\to 0} u'(q) = \infty$ and $\lim_{q\to 0}\varepsilon \in (0,\infty)$ it follows that $\lim_{q\to 0} u(q)/q = \infty$. By L'Hospital's rule, $\lim_{q\to 0}\left(u(\kappa q)/\kappa q\right)/\left(u(q)/q\right) = \lim_{q\to 0} u'(\kappa q)/u'(q)$ for all κ so the condition (7) in holds because $\kappa \neq \tilde{\kappa}$ implies $\lim_{q\to 0} u'(\kappa q)/u'(q) \neq \lim_{q\to 0} u'(\tilde{\kappa} q)/u'(q)$. □

Lemma. *At extreme quantities, social and private markups align as follows:*

1. *If* $\lim_{q\to 0} 1-\varepsilon(q) < 1$ *then* $\lim_{q\to 0} 1-\varepsilon(q) = \lim_{q\to 0}\mu(q)$.
2. *If* $\lim_{q\to\infty} 1-\varepsilon(q) < 1$ *then* $\lim_{q\to\infty} 1-\varepsilon(q) = \lim_{q\to\infty}\mu(q)$.

Proof. By assumption, $\lim_{q\to 0}\varepsilon(q) > 0$. Expanding this limit via L'Hospital's rule shows

$$\begin{aligned}
\lim_{q\to 0}\varepsilon(q) = \lim_{q\to 0} q/\left(u(q)/u'(q)\right) \qquad &= \lim_{q\to 0} 1/\lim_{q\to 0}\left(1 - u(q)u''(q)/(u'(q))^2\right) \\
= 1/\lim_{q\to 0}\left(1+\mu(q)/\varepsilon(q)\right) &= \lim_{q\to 0}\varepsilon(q)/\lim_{q\to 0}\left(\varepsilon(q)+\mu(q)\right)
\end{aligned}$$

which gives the first part of the result. Identical steps for $q \longrightarrow \infty$ give the second part. □

5 Robustness of Efficiency under CES Demand

5.1 Multiple Sectors

The optimal and market allocations can be summarized as solutions to the following maximization problem with $\alpha = 0$ and $\alpha = 1$ respectively:

$$\max_{q_0, M_e, c_d, q(c)} U\left(q_0, M_e \int_0^{c_d} v_\alpha(q(c))\, dG\right) \text{ subject to } 1 \geq q_0 + M_e \left(\int_0^{c_d} (cq(c) + f/L)\, dG + f_e/L\right). \tag{8}$$

where $v_\alpha(q) \equiv \alpha u'(q)\, q + (1-\alpha)\, u(q)$. Conditional on a resource allocation of $(1-q_0)$ towards differentiated goods, the FOCs for the optimal and market allocations within the differentiated goods sector are similar to the one-sector case. Therefore, the distortion results for the one-sector case hold for the multi-sector case, given a fixed resource allocation $(1-q_0)$.

5.2 Non-constant Marginal Costs

To account for non-constant marginal costs, let the variable cost of production be $\omega(q) \cdot cq$. The benchmark model implicitly assumes $\omega(q) = 1$. Assuming strict concavity of the firm and the planner problems, the optimal and market allocations can be summarized as solutions to the following maximization program for $\alpha = 0$ and $\alpha = 1$. Using the constructed function $v_\alpha(q) \equiv \alpha u'(q)\, q + (1-\alpha)\, u(q)$,

$$\max_{M_e, c_d, q(c)} L M_e \int_0^{c_d} v_\alpha(q(c))\, dG \text{ subject to } L \geq M_e \left(\int_0^{c_d} L\omega(q(c)) c q(c) + f dG + f_e\right). \tag{9}$$

The FOCs for $q(c)$ and c_d are:

$$\begin{aligned} u' + \alpha u'' q =& \beta c\left[\omega + \omega' q\right] \\ u_d + \alpha\left(u'_d q_d - u_d\right) =& \beta\left[\omega_d c_d q_d + f/L\right] \end{aligned}$$

where β is the Lagrange multiplier. The FOC for M_e is unchanged and the labor constraint is as specified above. From the envelope theorem, $d\ln\beta/d\ln\alpha = 1 - \int u / \int v$. Totally differentiating the c_d FOC and substituting for $d\ln\beta/d\ln\alpha$, the change in the cost cutoff is

$$d\ln c_d/d\ln\alpha = \frac{\omega_d c_d q_d + f/L}{\omega_d c_d q_d} \frac{\alpha\left[(1-\bar{\varepsilon}) - (1-\varepsilon_d)\right]}{\left[1-\alpha(1-\bar{\varepsilon})\right]\left[1-\alpha(1-\varepsilon_d)\right]}.$$

Therefore, sign $d\ln c_d/d\ln\alpha =$ sign $(1-\varepsilon)'$ and introducing non-constant marginal costs does not change the comparative static for firm selection. The market under-selects when $(1-\varepsilon)' > 0$ and vice-versa.

The quantity allocation can be determined through the following relationship:

$$\left(1-\mu(q_{\text{mkt}}(c))\right)\left[\left(\frac{u'}{\omega+\omega'q}\right)_{\text{mkt}}/\left(\frac{u'}{\omega+\omega'q}\right)_{\text{opt}}\right]=\delta/\lambda$$

Under the earlier assumption of constant marginal costs, $\omega+\omega'q=1$ and the relevant relationship is $\left(1-\mu(q_{\text{mkt}}(c))\right)\left[u'(q_{\text{mkt}})/u'(q_{\text{opt}})\right]=\delta/\lambda$. The key difference under non-constant marginal costs is that we must now examine $h(q)\equiv u'(q)/(\omega(q)+\omega'(q)q)$. Under the concavity assumption ($2\omega'+\omega''q>0$), $h'(q)<0$ and $d\ln q/d\ln c<0$. Therefore, the logic of the quantity distortion proof continues to hold.

5.3 Advertising Costs

We assume advertising costs are such that the concavity of the problem is maintained in conjunction with VES demand. It may be shown that revenue maximization holds under advertising costs. Defining $\bar{\varepsilon}\equiv\int(v/u)\cdot\frac{un}{\int un}=\int vn/\int un$, we can proceed as earlier to determine the distortions in market outcomes.

Proof of Quantity Distortion. The change in quantity with respect to α is $\text{sign}d\ln q/d\ln\alpha=(-)\text{sign}\left[(\mu-\bar{\mu})-(1-\bar{\varepsilon}-\bar{\mu})\right]$. For $\mu',(1-\varepsilon)'>0$, the sign is positive for all $c<\bar{c}$ (where $\bar{c}$ is such that $\mu(\bar{c})=\bar{\mu}$ for weighting $un/\int undG$). For $c>\bar{c}$, $\mu<\bar{\mu}$ but the second term $1-\bar{\varepsilon}-\bar{\mu}<0$. As μ is increasing, we will check whether the sign can be negative for $c\to\infty$. Using $\sup(1-\mu)=\sup\varepsilon$, the term in square brackets at $c\to\infty$ is $\bar{\varepsilon}-\sup\varepsilon<0$. Therefore, the market over-produces quantity for low c and under-produces for high c. For $\mu',(1-\varepsilon)'<0$, the argument is reversed because at $c\to\infty$, the term is $\bar{\varepsilon}-\inf\varepsilon>0$. For the misaligned case of $\mu'>0,(1-\varepsilon)'<0$, the sign is negative for $c>\bar{c}$. For $c<\bar{c}$, the first term and the second term are both positive, so we check the sign at $c\to 0$. At zero, using $\inf_{c\leq c_d^{\text{mkt}}}1-\mu\left(q^{\text{mkt}}(c)\right)\geq\overline{\sigma}$ and $\overline{\sigma}\equiv\sup_{c\leq c_d^{\text{mkt}}}\varepsilon\left(q^{\text{mkt}}(c)\right)$, the term in square brackets is $(\mu-\bar{\mu})-(1-\bar{\varepsilon}-\bar{\mu})\leq\bar{\varepsilon}-\bar{\sigma}=\bar{\varepsilon}-\sup\varepsilon\leq 0$. This implies the sign is negative for all c and the market over-produces quantity. When $\mu'<0,(1-\varepsilon)'>0$, the sign is positive for $c>\bar{c}$. For $c<\bar{c}$, the first term and the second term are both negative, so we check the sign at $c\to 0$. At zero, using $\sup_{c\leq c_d^{\text{mkt}}}1-\mu\left(q^{\text{mkt}}(c)\right)\leq\underline{\sigma}$ and $\inf_{c\leq c_d^{\text{opt}}}\varepsilon\left(q^{\text{opt}}(c)\right)\equiv\underline{\sigma}$, the term in square brackets is $(\mu-\bar{\mu})-(1-\bar{\varepsilon}-\bar{\mu})\geq\bar{\varepsilon}-\underline{\sigma}=\bar{\varepsilon}-\inf\varepsilon\geq 0$. This implies the sign is positive for all c and the market under-produces quantity. We conclude that the quantity misallocation is similar to the baseline model.

Proof of Productivity. As earlier, $\text{sign}d\ln c_d/d\ln\alpha=\text{sign}(1-\varepsilon)'$. Since $dc_d/d\alpha\gtrless 0$ when $(1-\varepsilon)'\gtrless 0$, the market under-selects for $(1-\varepsilon)'>0$ and over-selects in the other case. The bias in firm selection is similar to that for the baseline model.

Proof of Advertising. The change in advertising is

$$\frac{d\ln n}{d\ln\alpha}=\frac{1-n}{\beta n}\frac{1}{1-v'_{\alpha}q/v_{\alpha}}\frac{\alpha\left((1-\bar{\varepsilon})-(1-\varepsilon)\right)}{(1-\alpha(1-\bar{\varepsilon}))\left(1-\alpha(1-\varepsilon)\right)}.$$

Therefore, $\text{sign}\, d\ln n/d\ln\alpha = \text{sign}\left((1-\bar{\varepsilon})-(1-\varepsilon)\right)$. When $(1-\varepsilon)' > 0$, the market under-advertises for low c and over-advertises for high c. For $(1-\varepsilon)' < 0$, the market over-advertises for low c and under-advertises for high c.

References

Arkolakis, C., A. Costinot, and A. Rodriguez-Clare, "New trade models, same old gains?," *American Economic Review*, 2012, *102* (1), 94–130.